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2D NUMERICAL MODEL OF CHANNEL BED FORMED OF RANDOMLY DISTRIBUTED SPHERICAL PARTICLES.

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Summary: *The paper deals with the saltation mode of bed load transport. The saltation process modelling consists of two parts – deterministic motion of the particle in fluid and stochastic process of its collision with bed, which is the stochastic part of a Monte-Carlo simulation. The channel bed is formed by spherical particles of non-uniform size, the distribution of the particles (in the collision place and time) is determined according to Gaussian dense function. The numerical method for a bed forming and determination of collision coordinates and a collision angle is proposed.*

1. Introduction.

Saltation is a predominant mode of bed load transport in rivers and channels. During saltation particles hop up from the channel bed and follow ballistic like trajectory till the next bounce with bed. Though the water flow in channel usually occurs in turbulent mode, the modeling of saltation deals with time averaged fluid velocity profile and turbulence is not taken into account (Nino & Garcia, 1994; Sekine & Kikkawa, 1992; Lee & Hsu, 1994; Van Rijn, 1984; Lee et al., 2000). The boundary conditions for the equations governing model particle motion are provided by the particle collision with the channel bed. Most of authors, who developed numerical models of particles saltation in open channel with rough bed, also developed their own stochastic collision-rebound models.

In 2D collision model of Nino & Garcia (1994) saltating particle velocities immediately after collision were estimated from particle velocities immediately before collision by introducing coefficients of restitution and friction, which reduce the incident particle velocity components in the normal and tangential directions, respectively, with respect to the contact surface at the bed. The contact surface is characterized by angle of collision, i.e. the angle between the bed plane and tangent to the saltating particle surface at a contact point. Nino & Garcia (1994) derived the angle of collision by geometrical transformations from the given incidence angle and random coordinates in geometrically determined range of point along a vertical line passing through the centre of bed particle. The random parameter was distributed with a uniform probability density. According to Nino & Garcia (1994), prescribing a uniform probability density for the model random parameter is equivalent to assuming that the bed particle has a uniform probability being located anywhere in the bed.

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Sekine & Kikkawa (1992) in their 3D model constructed a bed on a whole distance of saltation. The bed was formed of spherical particles of equal diameter. The distribution of the bed particles along vertical line was of Gaussian type. Along horizontal plane the particles were located basing on the assumption of cubical packing. That is the particle centroid Cartesian coordinates x_i', y_j' were dependent linearly on particle indices i, j, respectively and could be calculated as: $x_i' = id_b$, $y_j' = jd_b$, d_b denotes bed surface particle diameter. The velocity components after collision were determined from the velocity components before collision by introducing coefficient of rebound which corresponded to the reduction of takeoff particle velocity components in the normal and two tangential directions with respect to the contact surface.

Lee & Hsu (1994) determined the boundary conditions simply assuming that the take-off velocity equals $2u_*$, and takeoff angle equals 45°, where u_* is the shear fluid velocity. Lee et al. (2000) in their investigation adopted the collision model of Nino & Garcia (1994).

Loukertchenko et al. (2002) developed 2D collision model, where angle and height of collision were stochastically calculated. The diameter of a bed particle was supposed to be a random variable. The bed particle, which the collision happened with, was located according to a uniform probability on a horizontal segment, which borders were determined by the saltating particle velocity vector just before collision, and geometrical considerations. The angular velocity and components of translational velocity immediately after the collision were calculated as functions of angular velocity and components of translational velocity immediately before the collision from momentum equations using restitution and friction coefficients.

Lukerchenko et al. (2004) developed 3D particle-bed stochastic collision model. The collision height and the contact point were defined as random variables. The zone of saltating particle surface, where the contact point could be located (so called the contact zone), was defined immediately before the collision. Then it was supposed that each point of the contact zone could be the contact point with equal probability. The particle velocities after the collision were calculated from the velocities before the collision.

As can be seen from the literature review, only Sekine & Kikkawa (1992) for the purposes of Direct Numerical Simulation (DNS) of saltation constructed the bed of equal particles, though the premise of bed particles cubical packing does not represent the real bed. The present 2D model is similar to the 3D model of Sekine & Kikkawa (1992) in that it uses the same distribution function of bed particle vertical coordinate, and differs from it in the following points. In the present paper the bed particles are of non-uniform diameter, a particle diameter is distributed according to Gaussian dense function. In model of Sekine & Kikkawa (1992) only vertical coordinate of a bed surface particle was found stochastically, the horizontal coordinates x_i', y_j' were found deterministically; in present model for the bed surface forming the random parameter along horizontal direction is used as well as the random parameters along vertical direction. In model of Sekine & Kikkawa (1992) the difference between adjacent bed particle x' and y' coordinates is equal and particles do not contact each other; in present model bed is constructed in agreement with the assumption of dense packing, i.e. bed surface particles contact each other and difference between x coordinates of adjacent particles is not fixed. The assumption of dense bed particles packing and the use of a stochastic numbers along horizontal line makes present model to seem to the author more realistic than the model of Sekine & Kikkawa (1992), though his model is 3-dimentional.

2. Coordinates and angle of collision.

The collision happens when the moving particle reaches the bed surface. Suppose that the collision occurs only at one point, so called the contact point, when the saltating particle contacts bed. To describe the collision it is necessary to determine the moment of collision, i.e. the coordinates of saltating particle at a moment of collision, and the angle of collision. Let us define bed surface particles as those bed particles that are seen from above. A saltating particle can contact only bed surface particle. The particles that are under the bed surface layer are out of interest for the saltation, and available statistical data about bed surface particles are sufficient for bed modeling. According to Sekine & Kikkawa (1992), in natural rivers bed-load transport often occurs over a bed composed of grains of similar size. Nino & Garcia (1998) investigated saltation over a fixed bed of sand particles with an approximately uniform size. In this research the radiuses of bed particles are distributed accordingly to Gaussian dense function with average $\overline{r_b}$ and standard deviation $\sigma_r \leq 0.3\overline{r_b}$.

Let us introduce a reference frame Oxy, where the axis x is directed along the bed surface downstream, and the axis y is orthogonal to the bed surface and directed upward. The bed particles are distributed uniformly along a line parallel to the bed surface. The distribution of visible bed surface particles along the axis y is of Gaussian form, and is independent of water stream shear velocity u_*, Sekine & Kikkawa (1992); the density function is illustrated in Fig. 1.

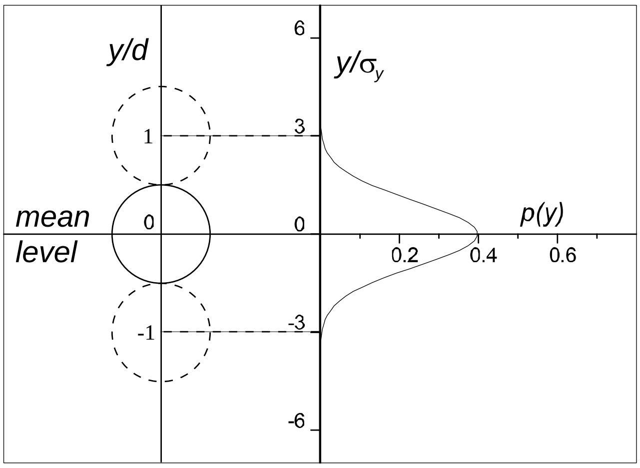


Figure 1 Plot of probability density associated with elevation of centroid of bed surface particle

According to Sekine & Kikkawa (1992), herein it is used their height distribution scheme with particle diameter in their scheme equal to the average bed particles diameter in present scheme. From Fig. 1 it follows that

$$\overline{d}_b = 3\sigma_y, \quad (1)$$

where $\overline{d}_b$ is average bed particle diameter, σ_y is standard deviation of elevation of bed particle centroid.

The bed is formed by the following way. The centroid y-coordinate y_{bi} and the radius r_{bi} of the particle with number i are determined stochastically according to Gaussian dense functions. After the radius r_{bi+1} and y-coordinate y_{bi+1} of the next particle are determined, the particle is placed closely to the previous. The x-coordinate x_{bi+1} is calculated as follows:

$$x_{bi+1} = x_{bi} + \sqrt{\left(r_{bi+1} + r_{bi}\right)^2 - \left(y_{bi+1} - y_{bi}\right)^2} \,. \tag{2}$$

The x_{bi+1} coordinate is calculated on the assumption that x_{bi} coordinate is known. The way of determining x_{b1} coordinate will be discussed below.

The Monte-Carlo simulation of a saltation involves the particle travel on a long distance that can reach several tenths of thousands of bed particle average diameter. The construction of a bed on the whole saltation distance in a saltation modeling could be wasting of the computational resources. Instead of this way here it is offered to construct a small part of the bed surface immediately below the moving particle just before the collision. The equations governing the particle motion are solved by an iterative scheme and the particle coordinates are known on each iteration step. When the lowest point of moving particle reaches the zone, where the probability of determining of bed particle surface is not negligibly small, see Fig. 1, the section of bed is constructed and the collision condition starts to be tested at every step of a program. The saltating particle is collided with the bed when the distance between the centroid of saltating particle and the centroid of any of the bed particles of the section is equal or lower then the sum of its radiuses, Fig. 2. That is, collision has occurred when

$$\left(x - x_{bi}\right)^2 + \left(y - y_{bi}\right)^2 \le \left(r + r_{bi}\right)^2 , \tag{3}$$

where x, y, r denote coordinates and radius of saltating particle, $x_{bi} y_{bi} r_{bi}$ denote coordinates and radius of bed particle with number i. Then the collision angle can be calculated as follows

$$\alpha = \mathrm{arctg}\left(\frac{x_{bi} - x}{y - y_{bi}}\right). \tag{4}$$

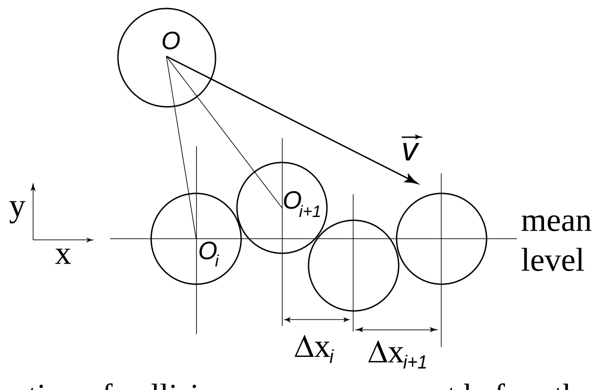


Figure 2 Schematic illustration of collision process, moment before the collision

The first bed particle of the section is placed in the area under the moving particle. Then the bed section can be constructed by placing bed particles up to the point of intersection of the bed particles mean level and vector of saltating particle velocity. The x-coordinate of the centroid of the first placed bed particle is determined accordingly to the uniform distribution on a line segment of length l located along x-axis, with the right border x-coordinate equal to the x-coordinate of the saltating particle centroid. The length of the segment l cannot be assigned arbitrarily. It should be equal to the mean difference between x-coordinates of the adjacent bed particles, see Fig. 2.

$$l = \overline{\Delta x_i}\,. \tag{5}$$

l is a constant value for known bed particle radius distribution, and can be calculated numerically in advance of simulating of a saltation. For purposes of calculating l the x-coordinate of first bed particle is not important, as only differences between x-coordinates are involved in the calculation. To explain equation (5) let us first consider a simpler model of bed packing.

If the bed is formed of spherical particles of equal diameter d_{eq} and y-coordinates of bed particles have a constant value, as illustrated in Fig. 3, then the distance between the centroid of the adjacent particles equals to d_{eq}. The probability density of determining any particle centroid at a point of the x-axis is uniform. The bed has only one degree of freedom – along the x-axis. The position of one particle determines the whole bed, if the position of the particle is shifted for the distance divisible by d_{eq} then it would determine the same bed as in position before shifting. The first particle position should be chosen on the segment of length equal to or divisible by diameter d_{eq}.

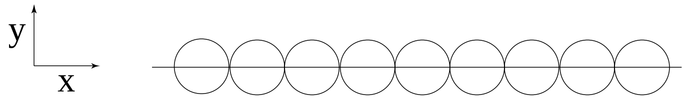


Figure 3 Simplified model of bed forming

Consider the case when the first particle centroid position is chosen on the segment not equal to the particle diameter. Suppose the first particle centroid is placed on the segment *AD* according to uniform distribution, see Fig. 4. If the particle falls into *AC* segment, then the neighboring right particle falls into the segment *BD*, and the probability density of finding any particle at the segment *BD* is twice as great as probability density of finding any particle at segment *CB*. By the same reasoning one can conclude that the probability density at segment AC is twice as great as probability density of finding any particle at segment *CB*, considering case when the first particle falls onto the segment *BD*. Then the probability density function of determining any bed particle centroid along x-axis would not be uniform; it would have

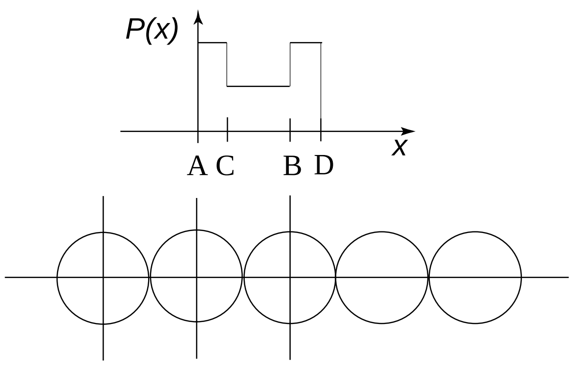


Figure 4 The probability density of determining any particle centroid at segment *AD* if the first particle centroid is placed at the segment *AD* according to the uniform probability law. $d_{eq} \le AD \le 2d_{eq}$; $AB = CD = d_{eq}$.

the shape similar to that shown at Fig. 4, and would be dependent on the coordinates of the borders of the line segment at which the first bed particle is placed.

In this paper the particle centers are not located on the line. As the particles are packed densely, the projection on the x-axis of the length between adjacent particle centers Δx_i is less or equal to the bed particle average diameter, see Fig. 2

$$\Delta x_i \leq \overline{d}_b . \tag{6}$$

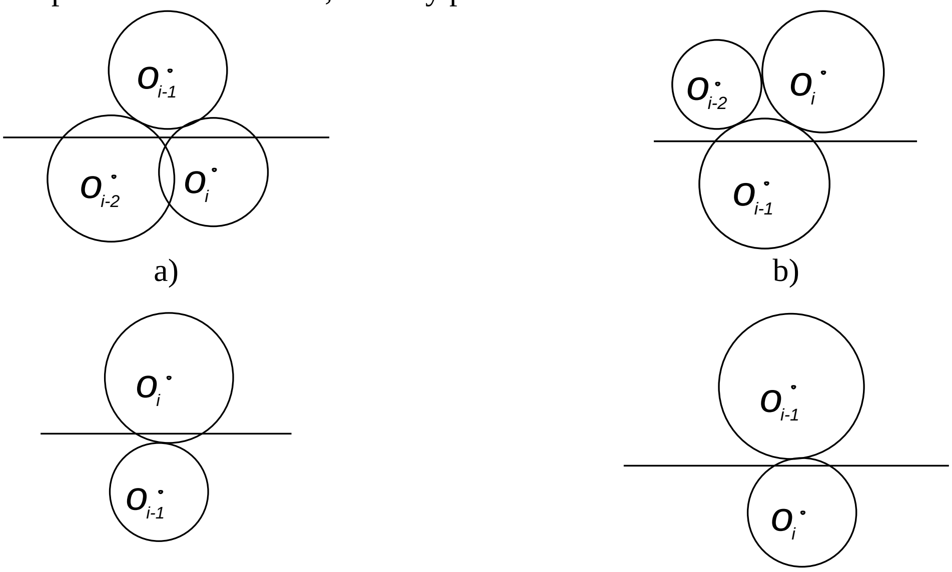
If we average the bed surface particles radiuses, and positions over x and y directions, and construct the imaginary bed, where fictitious bed particles will be located at their averaged positions, then the bed structure will be similar to that illustrated on Fig. 3, but the adjacent particles will overlap each other. The distance between the centres of the adjacent particles would be equal $\overline{\Delta x_i}$. Presuming that the above conclusions on simple model are valid for the present more complex model we will come to a conclusion that equation (5) is a correct statement.

By means of Direct Numerical Simulation the bed of 10000 particles was constructed and the average difference between x-coordinates of adjacent particles centers was found for different σ_r, see Table 1.

Table 1 The bed parameter versus standard deviation of the bed particle radius

$\sigma_r / \overline{r}_b$	$l / \overline{d}_b$
0.0	0.90
0.1	0.90
0.2	0.90
0.25	0.91
0.3	0.92

An auxiliary conditions used during the bed constructing were: the particle number i must not overlap the particle number $i-2$, and any particle must be seen from above.



c) d)

Figure 5 Cases eliminated from bed modeling

The first condition is illustrated in Fig. 5 (a), the second condition turns into two cases. When the particle i (together or without the particle number $i-2$) shadows the particle number $i-1$, see Fig. 5 (b, c); and when the particle is in the shadow of previous, see Fig. 5

(d). The probability of such case appearing is negligibly small and all such particles were eliminated from consideration. Also the cases when the particle could not reach the previous, $\left|y_{bi} - y_{b\,i-1}\right| > r_{bi} + r_{b\,i-1}$, were eliminated. The example of numerically modelled bed section is shown in Fig. 6.

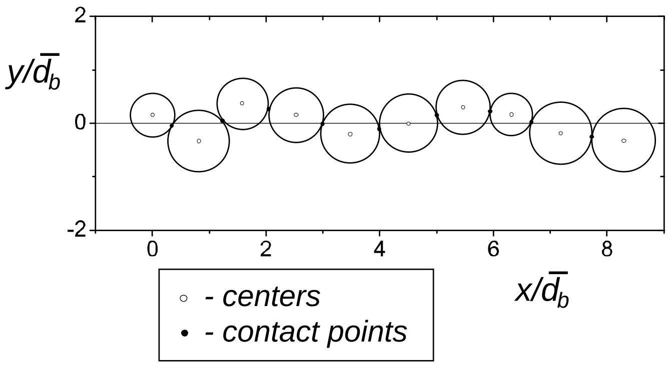


Figure 6 Example of modeled bed surface segment, $\sigma_r = 0.3\overline{r_b}$

3. Impulse equations

For the known contact point, vectors of angular and translational velocities immediately before the collision it is possible to calculate the vectors of translational velocity and rotational velocity immediately after the collision. As the model considered is two-dimensional, the vector of angular velocity has the only component in the direction orthogonal to the plane of motion ω. Let us introduce the reference frame $C\tau n$ with origin in the contact point, τ and n directions as shown in Fig. 7.

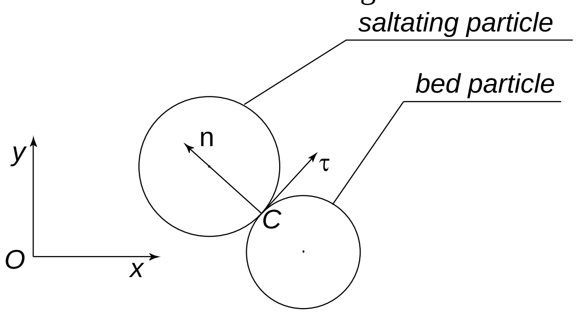


Figure 7 The reference frame associated with contact point

According to Loukertchenko et al. (2002), at the moment of collision for known collision angle the saltating particle velocity components immediately after the collision v_n^+, v_τ^+, and ω^+ can be expressed in terms of the particle velocity components immediately before the collision from following system of equations:

$$\begin{cases} (m+\gamma\lambda)\left(v_\tau^+ - v_\tau^-\right) = I_\tau, \quad J\left(\omega^+ - \omega^-\right) = rI_\tau \\ (m+\gamma\lambda)\left(v_n^+ - v_n^-\right) = I_n, \qquad v_n^+ = -ev_n^- \end{cases} \tag{7}$$

where r is radius of the particle, m is mass of the particle, e is restitution coefficient (situated between 0 and 1, depends solely on the materials of the particle and of the bed surface), J is particle moment of inertia, λ is added mass, γ is coefficient expressing influence of bed surface nearness on the particle added mass, I_τ, I_n denote momentum components transferred to the particle in τ and n direction, respectively; index “ – ” marks values before collision.

Let $v_{\tau C}$ denote the particle contact point τ -component of velocity, f_k denote coefficient of kinetic friction, and f_s denote coefficient of static friction. Equations (7) coupled with sticking condition $v_{\tau C}^{+} = v_{\tau}^{+} + r\omega^{+} = 0$ for case without slip, or with friction of law $\frac{I_\tau}{I_n} = -\operatorname{sign}\left(v_{\tau C}^{-}\right) f_k$ for slip case, form complete system of equations, that can be easily solved. The collision without slip is possible under the condition

$$\left| \frac{v_{\tau C}^{-}}{v_n^{-}} \right| \leq \frac{f_s\left(1+e\right)}{aJ}, \tag{8}$$

where $a = \left(J + \left(m + \gamma\lambda\right) r^2\right)^{-1}$.

Angular velocity and translational velocity components are expressed as follows

$$\begin{aligned} v_\tau^{+} &= v_\tau^{-} + \frac{I_\tau}{m + \gamma\lambda} \\ v_n^{+} &= v_n^{-} + \frac{I_n}{m + \gamma\lambda}, \\ \omega^{+} &= \omega^{-} + \frac{rI_\tau}{J} \end{aligned} \tag{9}$$

where

$$\begin{aligned} I_\tau &= -aJ\left(m + \gamma\lambda\right) v_{\tau C}^{-} \\ I_n &= -\left(1+e\right)\left(m + \gamma\lambda\right) v_n^{-} \end{aligned}, \tag{10}$$

in case of impact without slip, and

$$\begin{aligned} I_\tau &= \operatorname{sign}\left(v_{\tau C}^{-}\right) f_k \left(1+e\right)\left(m + \gamma\lambda\right) v_n^{-} \\ I_n &= -\left(1+e\right)\left(m + \gamma\lambda\right) v_n^{-} \end{aligned}, \tag{11}$$

in case of impact with slip. The velocity components in reference frame Oxy can be obviously obtained.

4. Conclusions

The collision-rebound model as a part of a saltation process modeling was developed. The emphasis was placed on the bed structure. For purposes of DNS the bed of spheres was modeled. The distribution of spheres radiuses is of Gaussian type. The distribution of spheres along vertical axis is also of Gaussian type, and is independent of water stream shear velocity u_*, Sekine & Kikkawa (1992). The distribution of bed particles along horizontal axis is uniform. The length of the line segment at which the first bed particle is placed accordingly to uniform dense function is determined for different values of σ_r.

The bed section is constructed immediately below the saltating particle just before the moment of collision when the lowest point of moving particle enters the zone $y \le 3\sigma_y$. The example of constructed bed section is illustrated in Fig. 6. The collision happens when the saltating particle reaches one of the bed particles, i.e. when the distance between the saltating particle and one of the constructed bed section particles becomes equal to the sum of their radiuses, see equation (3); after the collision happened the collision angle is calculated, see equation (4). To complete the collision model the impulse equations for rebound are cited and the components of particle velocity after bounce are expressed as functions of particle velocity components just immediately before collision, see (9) - (11).

5. Acknowledgements

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6. Notation

a - dimensionless parameter;
d_b - bed particle diameter;
d_{eq} - bed particle diameter in simplified case;
e - restitution coefficient;
f_k - coefficient of kinetic friction;
f_s - coefficient of static friction;
I_n - n-component of momentum transferred to saltating particle;
I_τ - τ -component of momentum transferred to saltating particle;
J - saltating particle moment of inertia;
l - length of line segment of the bed, where the first segment particle is placed;
m - mass of saltating particle;
r - saltating particle diameter;
r_{bi} - radius of bed particle with number i;
u_* - shear fluid velocity;
v_n - n-component of saltating particle velocity;
v_τ - τ -component of saltating particle velocity;
$v_{\tau C}$ - τ -component of saltating particle contact point velocity;
x_{bi} - x-coordinate of bed particle number i;
y_{bi} - y-coordinate of bed particle number i;
Δx_i - difference between x-coordinates of two adjacent bed particles;
λ - added mass of saltating particle;
σ_r - standard deviation of bed particle radiuses;
σ_y - standard deviation of bed particle centroid elevations over mean level;
ω - angular velocity of saltating particle.

Indices:
+ - after collision;
– - before collision;
overline - averaging.

7. References

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