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CGA-BASED SNAKE ROBOT CONTROL MODEL

Byrtus R.[*]

Abstract: *The snake robot is a nonholonomic mechanism composed of links equipped with passive wheels, connected by actuated joints whose motion mimics the locomotion of biological snakes. Control models intended for small-time local controllability are usually obtained by means of differential geometry, or, more recently, geometric algebra. Geometric algebras, also known as Clifford algebras, are an algebraic structure useful for modelling geometric objects and their transformations. We present an approach utilising the two-dimensional Conformal Geometric Algebra in order to derive the differential kinematics of the mechanism. A control model for small-time local controllability is created based on obtained differential kinematics, which is then used in visualization.*

Keywords: Geometric algebra, differential kinematics, snake robot, control model, nonholonomic mechanism.

1. Introduction

Let $\mathbb{R}^{3,1}$ be a vector space of dimension 4, with the orthogonal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_+, \boldsymbol{e}_-\}$ and an inner product such that $\boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = \boldsymbol{e}_+^2 = 1$ and $\boldsymbol{e}_-^2 = -1$. Define a new orthogonal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_0, \boldsymbol{e}_\infty\}$ with $\boldsymbol{e}_\infty = \boldsymbol{e}_- - \boldsymbol{e}_+$ and $\boldsymbol{e}_0 = \frac{1}{2}(\boldsymbol{e}_- + \boldsymbol{e}_+)$. The 2D Conformal Geometric Algebra (CGA) is the Clifford algebra $\mathbb{G}_{3,1}$, also denoted as $Cl_{3,1}$, with the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_0, \boldsymbol{e}_\infty\}$ along with the embedding Q of a point $[x, y] \in \mathbb{R}^2$ given by

$$Q : \mathbb{R}^2 \ni [x, y] \mapsto x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + \frac{1}{2}(x^2 + y^2)\boldsymbol{e}_\infty + \boldsymbol{e}_0. \tag{1}$$

Denote the product on $\mathbb{G}_{3,1}$ as $\circ : \mathbb{G}_{3,1} \times \mathbb{G}_{3,1} \to \mathbb{G}_{3,1}$ (later on, we will omit $\circ$ for brevity). An important property that defines the structure of the algebra is that for any vector $\boldsymbol{a} \in \mathbb{R}^{3,1}$, its geometric product coincides with its inner product: $\boldsymbol{a} \circ \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{a}$. An element of $\mathbb{G}_{3,1}$ is called a *multivector*. $\mathbb{G}_{3,1}$ along with multivector addition and scalar multiplication has the structure of a vector space. The operation $\circ$ is associative and distributive. It is not the only product we can define in $\mathbb{G}_{3,1}$, in fact, for vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^{3,1}$, it holds that

$$\boldsymbol{a} \circ \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}, \tag{2}$$

where $\cdot$ is the inner product defined earlier and $\wedge$ is the outer product. The inner and outer products can be extended to an arbitrary multivector (note that the eq. (2) does not hold in general for any two multivectors). The linear combination of geometric products of k linearly independent basis vectors is called a k-vector; for example, $\boldsymbol{e}_1 \circ \boldsymbol{e}_2$ is a 2-vector (also called a bivector).

The advantage of utilising this algebraic structure lies in its connection to geometry, as the name implies. It can be shown that using the inner and outer products, we are able to express geometric entities from the embedded space (in this case $\mathbb{R}^2$) as the null-spaces of embedded points w.r.t. the respective product used. For the outer product, we have the so-called Geometric Outer Product Null Space (GOPNS) and for the inner product, the Geometric Inner Product Null Space (GIPNS); usually, the G is omited, however it must be noted that in some literature, a distinction between IPNS and GIPNS (or OPNS and GOPNS) is made.

[*] Ing. Roman Byrtus: Institute of Mathematics, Brno University of Technology, Technická 2896; 616 69, Brno; CZ, roman.byrtus@vutbr.cz

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The geometric objects from $\mathbb{R}^2$ representable in CGA relevant to the scope of this paper are the embedded point $A_i = Q(x\boldsymbol{e}_1 + y\boldsymbol{e}_2)$, the point pair $P_i = A \wedge B$ given by the outer product of two embedded points A, B and the line $L = A \wedge B \wedge \boldsymbol{e}_\infty$ given by the wedge of points A, B and the null vector $\boldsymbol{e}_\infty$. In addition to geometric objects, there are elements of CGA that also represent transformations acting on these objects. In particular, let us introduce the *translator* T and the *rotor* R. The translator T represents translation in the direction of a vector $\boldsymbol{t} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2$ and in exponential notation can be written as

$$T = e^{-\frac{1}{2}\boldsymbol{t}\boldsymbol{e}_\infty}. \tag{3}$$

Similarly, the rotor R representing rotation around an axis L (given by a unit bivector, for example, the axis representing rotation around the origin in $\mathbb{R}^2$ is given by $\boldsymbol{e}_{12}$) by an angle α is expressed in exponential notation as

$$R = e^{-\frac{1}{2}\alpha L}. \tag{4}$$

The transformations are applied to a multivector using the sandwich product. For example, for a rotation around the origin of the line L by angle α, the rotated line L_{rot} is given by $L_{rot} = RL\tilde{R}$, where $\tilde{R} = \boldsymbol{e}^{\frac{1}{2}\alpha L}$ is the *reverse* of R.

2. 2D Snake Robot Model

The snake robot consists of a series of links of length $2l$, connected by actuated joints, in our case revolute joints. Denote the configuration space of the mechanism as the manifold $Q \subset (\mathbb{R}^2 \times (S^1)^3)$ with point $\boldsymbol{q} = [x, y, \theta, \phi_1, \phi_2]$ representing a configuration of the mechanism at the time t, see Fig. 1. We thus track the coordinates of a head point (x, y), a global angle of orientation θ and relative rotation angles between links ϕ_1, ϕ_2. The centre of every link is given by the point $p_i = (x_i, y_i)$, where passive wheels are attached.

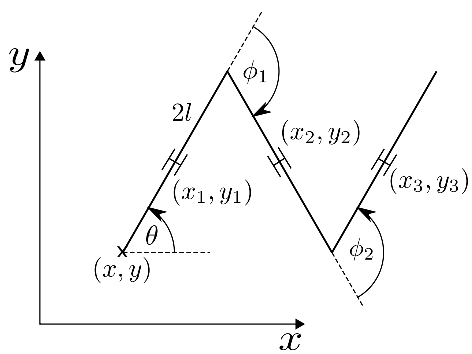

Fig. 1: A configuration of the mechanism.

The derivation of the control model is similar to the description given in Hrdina et al. (2016), but with the generalised transformations, we are able to model different mechanisms. The initial configuration of the i–th link of the mechanism is represented by point pairs $P_i^0 = A_i \wedge A_{i+1}$, where A_i are the edges of the links, see Fig. 2. We represent a general transformation M defined by bivector $L = L(\boldsymbol{q}(t))$ (depending state q at time t) as

$$M = e^{-\frac{1}{2}L(\boldsymbol{q}(t))},$$

and thus the reverse of M is $\tilde{M} = e^{\frac{1}{2}L(\boldsymbol{q}(t))}$.

A general configuration is then represented as a sequence of transformations applied to the initial configuration. Then the configuration of the i–th link at time t is given by

$$P_i = \prod_{j=k}^{1} M_j P_i^0 \prod_{j=1}^{k} \tilde{M}_j = T_{x,y} \prod_{j=i}^{1} R_j P_i^0 \prod_{j=1}^{i} (\tilde{R}_j) \tilde{T}_{x,y}, , \tag{5}$$

where M_j is the j–th transformation, $T_{x,y}$ is the translator from the origin to the head point and R_1 is the rotor representing the rotation w.r.t. global coordinate axes θ and the rotors R_2, R_3 represent the relative rotations ϕ_1, ϕ_2.

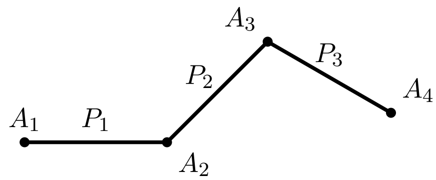

Fig. 2: A three-link snake robot represented in CGA.

To obtain the differential kinematics, we need to express the velocities of the state variables defining the mechanism's configuration, that is $\dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}_1$ and $\dot{\phi}_2$. The constraint imposed on snake robots is the non-slip nonholonomic constraint, which limits the velocity of the i–th link to the direction defined by the point pair P_i. In terms of CGA, we can express this constraint as

$$\dot{p}_i \wedge P_i \wedge \boldsymbol{e}_\infty = 0 \tag{6}$$

where $\dot{p}_i$ is the velocity of the i–th point pair's centre p_i. In fact, this condition expresses the geometric fact that the velocity $\dot{p}_i$ coincides with the line $P_i \wedge \boldsymbol{e}_\infty$ passing through the point pair P_i. The centre p_i is obtained by the decomposition

$$p_i = P_i \boldsymbol{e}_\infty \tilde{P}_i. \tag{7}$$

Taking the derivative w.r.t. time of eq. (7), we get

$$\dot{p}_i = \partial_t(P_i \boldsymbol{e}_\infty \tilde{P}_i) = \dot{P}_i \boldsymbol{e}_\infty \tilde{P}_i + P_i \boldsymbol{e}_\infty \dot{\tilde{P}}_i. \tag{8}$$

Assuming the state of P_i is represented by k transformations, expressing $\dot{P}_i$ we arrive to

$$\dot{P}_i = \partial_t(\prod_{j=k}^{1} M_j P_i^0 \prod_{j=1}^{k} \tilde{M}_j). \tag{9}$$

The derivative of the general transformation M is then given by

$$\partial_t M = -\frac{1}{2}(\partial_t L(\boldsymbol{q}(t)))e^{-\frac{1}{2}L(\boldsymbol{q}(t))} = -\frac{1}{2}\dot{L}(\boldsymbol{q}(t))M \tag{10}$$

and thus the derivative of the reverse is $\partial_t \tilde{M} = \frac{1}{2}\dot{L}\tilde{M}$. By chain rule

$$\dot{L} = \partial_t L(\boldsymbol{q}(t)) = \sum_{i=1}^{n}(\partial_{q_i} L)\dot{\boldsymbol{q}}_i. \tag{11}$$

Denoting $\partial_t M = \dot{M}$ and expanding eq. (9), we get

$$\dot{P}_i = \partial_t(\prod_{j=k}^{1} M_j P_i^0 \prod_{j=1}^{k} \tilde{M}_j) = \sum_{j=1}^{k}[P_i \cdot \dot{L}_j], \tag{12}$$

utilising Lemma 1 from Hrdina and Vašík (2015) in the last step, with $[P_i \cdot \dot{L}_j] = P_i \cdot L_j - L_j \cdot P_i$ being the commutator w.r.t. the inner product. Substituting eq. (12) into eq. (8) we can write $\dot{p}_i$ in the form of

$$\dot{p}_i = \sum_{j=1}^{k}[p_i \cdot \dot{L}_j]. \tag{13}$$

Finally, substituting Eq. (5) and Eq. (13) into the nonholonomic condition Eq. (6), we arrive to a set of three differential equations with multivector coefficients:

$$\begin{aligned}
&\Big(\dot{\theta} - 2\dot{x}\sin(\theta) + 2\dot{y}\cos(\theta)\Big)\boldsymbol{I} = 0,\\
&\Big(\dot{\phi}_1 + 2\dot{\theta}\cos(\phi_1) + \dot{\theta} - 2\dot{x}\sin(\phi_1+\theta) + 2\dot{y}\cos(\phi_1+\theta)\Big)\boldsymbol{I} = 0,\\
&\Big(2\dot{\phi}_1\cos(\phi_2) + \dot{\phi}_1 + \dot{\phi}_2 + 2\dot{\theta}\cos(\phi_2) + 2\dot{\theta}\cos(\phi_1+\phi_2) + \dot{\theta} -\\
&-2\dot{x}\sin(\phi_1+\phi_2+\theta) + 2\dot{y}\cos(\phi_1+\phi_2+\theta)\big)\boldsymbol{I} = 0,
\end{aligned} \tag{14}$$

where $\boldsymbol{I} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_0\boldsymbol{e}_\infty$ is the pseudoscalar. Since $\boldsymbol{I}$ is nonzero, it holds that the coefficient of the pseudoscalar must be zero, and thus we have arrived to the dynamical system describing the differential kinematics of the 3–link robotic snake. We have obtained only three equations, meaning two more equations will have to be added in order to define the control system for the mechanism. Denote $u_1 = u_1(t)$, $u_2 = u_2(t)$ as the control inputs. Then by adding two equations $\dot{\phi}_1 = u_1$, $\dot{\phi}_2 = u_2$, the forward kinematics are obtained. Note that the forward kinematics would be obtained by adding the equations $\dot{x} = u_1$, $\dot{y} = u_2$ instead. The final control system can be represented in vector form as

$$\dot{q} = X_1 u_1 + X_2 u_2, \tag{15}$$

where X_1, X_2 are control vector fields obtained by expressing $\dot{x}, \dot{y}, \dot{\theta}$ from eq. (14) along with the added control inputs u_1, u_2. Setting an initial configuration $\boldsymbol{q}_0$ as $\boldsymbol{q}_0 = [0, 0, 0, -\frac{\pi}{3}, \frac{\pi}{3}]$ and the controls as $u_1(t) = u_2(t) = 1$, the resulting motion can be seen in Fig. 3.

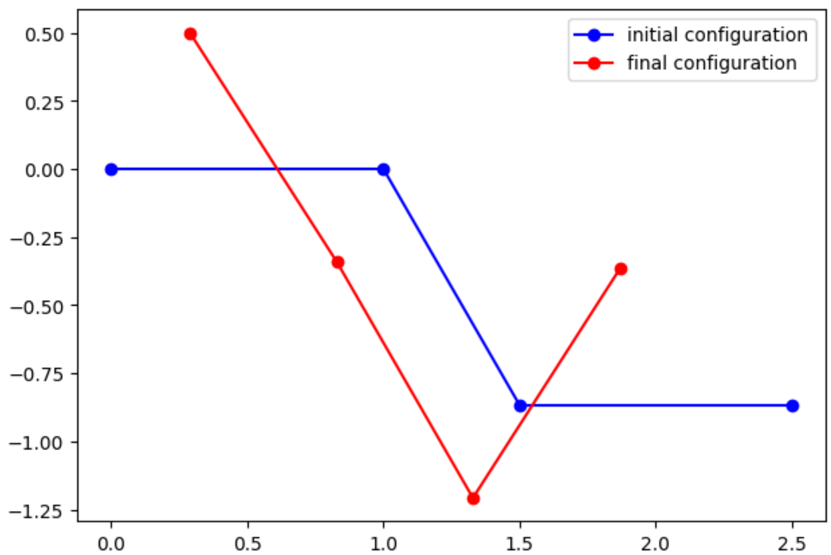

Fig. 3: The movement from the initial state in blue $\boldsymbol{q}_0$ into the final state $\boldsymbol{q}_f$ in red.

3. Conclusion

The application of geometric algebra in tasks involving geometry leads to a much more intuitive description of the underlying problems. Another advantage of using geometric algebra is that models are easily extended into higher dimensions - for example, in order to obtain the 3D planar locomotion model, it would be enough to add an extra dimension representing the z-axis, which leads to the 3D CGA. The extra dimension would appear in the relevant places (representation of the configuration, transformations having an extra dimension, etc.), but the formulas remain the same.

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