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PROBABILITY LIMITS OF THE CRITICAL ROTOR SPEEDS

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Abstract: *Rotors are manufactured with certain tolerances, implying that both outer and inner diameters of the rotor can be treated as random parameters (RP). This paper focuses on an approach for estimating the mean and variance of the critical speeds of rotors. These values allow for probabilistic determination of the upper and lower limits of critical speeds. The mean of critical speeds has to be derived iteratively due to the frequency (revolution) dependence of gyroscopic and circulation matrices. Subsequently, the critical speed is approximated using two terms of Taylor's expansion at the mean value of critical speed. It is essential to conduct sensitivity analysis of the critical speed concerning RP. This approximate approach avoids the necessity of knowing the probability function of randomly valued diameters, respecting the validity of Chebyshev's inequality. Rotor discretization in this study is achieved using the Finite Element Method.*

Keywords: Critical speed, probability, FEM, sensitivity analysis, rotor dynamics.

1. Introduction

The discretized mathematical model of the rotating rotor without external excitation can be represented by well known equation of motion (system of the *n* differential equations of the 2nd order) (e.g. Dimarogonas, (1996))

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\left[\mathbf{B}+\omega\mathbf{G}\right]\dot{\mathbf{q}}(t)+\left[\mathbf{K}+\omega\mathbf{K}_c\right]\mathbf{q}(t)=\mathbf{0}. \tag{1}$$

where **M, B, G, K,** $\mathbf{K}_c$ is matrix of mass, damping, gyroscopic effect, stiffness and circulation, respectively, all of order *n*. The quantity $\mathbf{q}(t)$ corresponds to vector of generalized displacements and its differentiations are marked by dots. Adding the trivial identity

$$\mathbf{M}\dot{\mathbf{q}}(t)-\mathbf{M}\dot{\mathbf{q}}(t)=\mathbf{0}, \tag{2}$$

to (1) we can come after simple rearrangements to the equation (system of the 2*n* differential equations of the 1st order)

$$\dot{\mathbf{u}}(t)=\mathbf{A}(\omega)\mathbf{u}(t), \tag{3}$$

where

$$\mathbf{A}(\omega)=\begin{bmatrix}\mathbf{0}, & \mathbf{I}\\ -\mathbf{M}^{-1}\left(\mathbf{K}+\omega\mathbf{K}_c\right), & -\mathbf{M}^{-1}\left(\mathbf{B}+\omega\mathbf{G}\right)\end{bmatrix},\quad \mathbf{u}(t)=\begin{bmatrix}\mathbf{q}(t)\\ \dot{\mathbf{q}}(t)\end{bmatrix}. \tag{4}$$

The Eq. (3) leads to the eigenvalue problem

$$\left[\mathbf{A}(\omega)-\lambda\mathbf{I}\right]\mathbf{v}=\mathbf{0}, \tag{5}$$

which can be solved for i^{th} critical speed in the iteration way respecting some starting chosen initial value by iteration way

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$$\begin{aligned}&\left[\mathbf{A}\left(\omega_i^{(r-1)}\right)-\lambda_i^{(r)}\mathbf{I}\right]\mathbf{v}_i^{(r)}=\mathbf{0}\ \Rightarrow\ \lambda_i^{(r)}=?\\&\omega_i^{(r)}=\operatorname{Im}\left\{\lambda_i^{(r)}\right\}\ \Rightarrow\ \omega_{crit,i}=\lim_{r\to\infty}\left\{\operatorname{Im}\left[\lambda_i^{(r)}\right]\right\}\end{aligned}\quad. \tag{6}$$

Each critical speed can be alternatively assessed by Campbell's diagram, too.

2. Probability approach

Let us assemble the independent input random parameters (RP) to the vector of RP and apply the expectation operator. We can come to the mean vector of RP

$$\boldsymbol{\mu}_{\mathbf{p}}=E\left\{\left[p_1,\ p_2,\ \ldots\ ,\ p_s\right]^T\right\}\in\mathbf{R}^{s,1}. \tag{7}$$

Let us introduce the critical eigenvalue vector in form

$$\boldsymbol{\lambda}_{crit}=\left[\lambda_1,\quad \lambda_2,\quad \ldots,\quad \lambda_b,\right]^T\in\mathbf{C}^{b,1} \tag{8}$$

real part of which corresponds to the vector of critical angular speeds. There is a well-known method to calculate derivatives of one eigenvalue e.g. $\lambda_j=\lambda_{crit,i}$ with respect to some parameter e.g. p_k. Let us derivate the equation

$$\left[\mathbf{A}\left(\omega_{crit,j}\right)-\lambda_j\mathbf{I}\right]\mathbf{v}_j=\mathbf{0} \tag{9}$$

with respect to parameter p_k and pre-multiply by $\mathbf{v}_j^{*T}$. Now we can write

$$\mathbf{v}_j^{*T}\frac{\partial\mathbf{A}\left(\omega_{crit,j}\right)}{\partial p_k}\mathbf{v}_j-\frac{\partial\lambda_i}{\partial p_k}\underbrace{\mathbf{v}_j^{*T}\mathbf{v}_j}_{1}+\underbrace{\mathbf{v}_j^{*T}\left[\mathbf{A}\left(\omega_{crit,j}\right)-\lambda_j\mathbf{I}\right]}_{\mathbf{0}}\frac{\partial\mathbf{v}_j}{\partial p_k}=0,$$

$$\Downarrow$$

$$\frac{\partial\lambda_j}{\partial p_k}=\mathbf{v}_j^{*T}\frac{\partial\mathbf{A}\left(\omega_{crit,j}\right)}{\partial p_k}\mathbf{v}_j. \tag{10}$$

Let us assemble the elements (10) into the sensitivity matrix

$$\frac{\partial\boldsymbol{\lambda}_{crit}}{\partial\mathbf{p}}=\left\{\frac{\partial\lambda_j}{\partial p_k}\right\}_{j,k}. \tag{11}$$

The relation between derivatives $\frac{\partial\lambda_j}{\partial p_k}$ and $\frac{\partial\Omega_{Dj}}{\partial p_k}$ can be obtained by ($\Omega_{Dj}=\Omega_{crit,j}=\omega_{crit,j}$)

$$\lambda_j=-D_j\Omega_j+i\underbrace{\Omega_j\sqrt{1-D_j^2}}_{\Omega_{Dj}}=\left(\frac{-D_j}{\sqrt{1-D_j^2}}+i\right)\Omega_{Dj}, \tag{12}$$

and then the derivatives of the j^{th} critical angular speed with respect to k^{th} RP has form

$$\frac{\partial\Omega_{Dj}}{\partial p_k}=\frac{\dfrac{\partial\lambda_j}{\partial p_k}}{\dfrac{\partial\lambda_j}{\partial\Omega_{Dj}}}=\frac{1}{\left(\dfrac{-D_j}{\sqrt{1-D_j^2}}+i\right)}\frac{\partial\lambda_j}{\partial p_k}\quad. \tag{13}$$

The eigenfrequencies of un-damped system and damping ratios can be obtained according to following relations (Ω_j corresponds to the *j*-th eigenfrequency of un-damped system)

$$\left.\begin{array}{r}\Omega_{Dj}=\mathrm{Im}\left\{\lambda_j\right\}=\Omega_j\sqrt{1-D_j^2}\\ -D_j\Omega_j=\mathrm{Re}\left\{\lambda_j\right\}\end{array}\right\}\Rightarrow D_j=\frac{-\mathrm{Re}\left\{\lambda_j\right\}}{\left|\lambda_j\right|},\quad \Omega_j=\left|\lambda_j\right|.$$

These quantities are depicted in Fig. 1

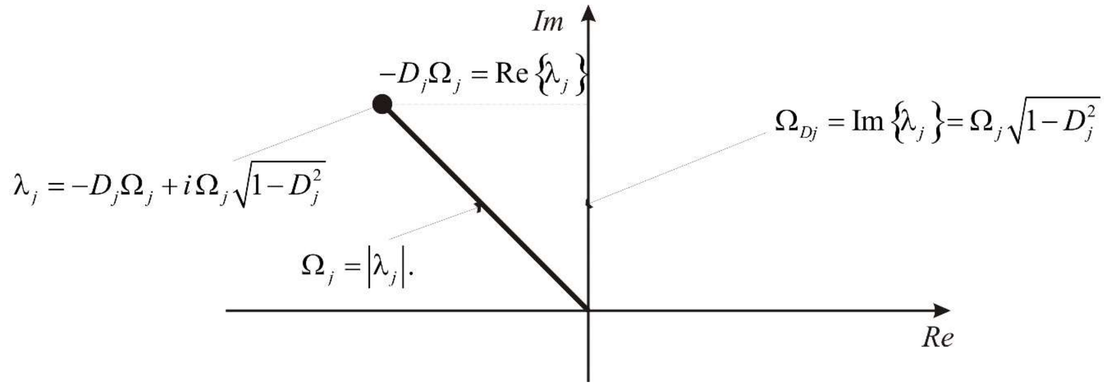

Fig. 1 Complex eigenvalues.

Let us introduce diagonal complex matrix $\mathbf{D}=diag\left\{\left(\frac{-D_j}{\sqrt{1-D_j^2}}+i\right)^{-1}\right\}$, enabling rewrite the Eq. (13) into the compact form

$$\frac{\partial\mathbf{\Omega}_D}{\partial\mathbf{p}}=\mathbf{D}\frac{\partial\boldsymbol{\lambda}_{crit}}{\partial\mathbf{p}}. \tag{14}$$

Let us express the vector of critical angular speeds by means of the first two terms of Taylor's series about mean vector of RP (Dupal, 2008). We can come to brief form (only the 1[st] derivatives)

$$\mathbf{\Omega}_D\doteq\mathbf{\Omega}_D\left(\boldsymbol{\mu}_\mathbf{p}\right)+\left.\frac{\partial\mathbf{\Omega}_D}{\partial\mathbf{p}}\right|_{\mathbf{p}=\boldsymbol{\mu}_\mathbf{p}}\left(\mathbf{p}-\boldsymbol{\mu}_\mathbf{p}\right)=\mathbf{\Omega}_D\left(\boldsymbol{\mu}_\mathbf{p}\right)+\frac{\partial\mathbf{\Omega}_D}{\partial\mathbf{p}}\left(\mathbf{p}-\boldsymbol{\mu}_\mathbf{p}\right). \tag{15}$$

The terms containing the 2[nd] derivatives and higher were left out respecting linear transformation relation between diameters of shafts and their eigenvalues. Applying expectation operator $E\left\{\mathbf{\Omega}_D\right\}$ to the Eq. (15) we obtain approximate relation for mean value of vector of critical angular speeds in form

$$\boldsymbol{\mu}_{\mathbf{\Omega}_D}\doteq\mathbf{\Omega}_D\left(\boldsymbol{\mu}_\mathbf{p}\right). \tag{16}$$

Applying variance operator $E\left\{\left(\mathbf{\Omega}_D-\boldsymbol{\mu}_{\mathbf{\Omega}_D}\right)\left(\mathbf{\Omega}_D-\boldsymbol{\mu}_{\mathbf{\Omega}_D}\right)^T\right\}$ to the Eq. (15) respecting (16) we can come to the covariation matrix of critical speed vector in form

$$\mathbf{\Sigma}_{\mathbf{\Omega}_D}=E\left\{\frac{\partial\mathbf{\Omega}_D}{\partial\mathbf{p}}\left(\mathbf{p}-\boldsymbol{\mu}_\mathbf{p}\right)\left(\mathbf{p}-\boldsymbol{\mu}_\mathbf{p}\right)^T\frac{\partial\mathbf{\Omega}_D^T}{\partial\mathbf{p}}\right\}=\frac{\partial\mathbf{\Omega}_D}{\partial\mathbf{p}}E\left\{\left(\mathbf{p}-\boldsymbol{\mu}_\mathbf{p}\right)\left(\mathbf{p}-\boldsymbol{\mu}_\mathbf{p}\right)^T\right\}\frac{\partial\mathbf{\Omega}_D^T}{\partial\mathbf{p}}=\frac{\partial\mathbf{\Omega}_D}{\partial\mathbf{p}}\mathbf{\Sigma}_\mathbf{p}\frac{\partial\mathbf{\Omega}_D^T}{\partial\mathbf{p}}. \tag{17}$$

The last relation can be rearranged by means of (14) into form

$$\mathbf{\Sigma}_{\mathbf{\Omega}_D}=\mathbf{D}\frac{\partial\boldsymbol{\lambda}_{crit}}{\partial\mathbf{p}}\mathbf{\Sigma}_\mathbf{p}\frac{\partial\boldsymbol{\lambda}_{crit}^T}{\partial\mathbf{p}}\mathbf{D}, \tag{18}$$

where $\mathbf{\Sigma}_\mathbf{p}$ is covariation matrix of input RP (when parameters are independent this matrix is diagonal).

3. Application

Random parameters: D_1, D_2 from the left hand side, probability density function-normal (Gauss), variances $\sigma_{D_i}^2 = 1e-7m, i = 1, 2.$ $\mathbf{\Sigma_p} = diag\{\sigma_{D_i}^2\}$. Mean values $\mu_{D_1} = 0.012\ m,\ \mu_{D_2} = 0.016\ m$.

Results: $\sigma_{\Omega_{D1}}^2 = 0.8149\ (rad/s)^2,\ \sigma_{\Omega_{D2}}^2 = 0.8332\ (rad/s)^2.$

Monte Carlo $\sigma_{\Omega_{D1}}^2 = 0.8149\ (rad/s)^2,\ \sigma_{\Omega_{D2}}^2 = 0.8390\ (rad/s)^2$

$P\{\mu_{\Omega_{Di}} + 3\sigma_{\Omega_{Di}} \le \Omega_{Di} \le \mu_{\Omega_{Di}} + 3\sigma_{\Omega_{Di}}\} = 0.9973$ (Gauss), Monte Carlo simulation relative frequency of the inequality satisfaction is 0.9972.

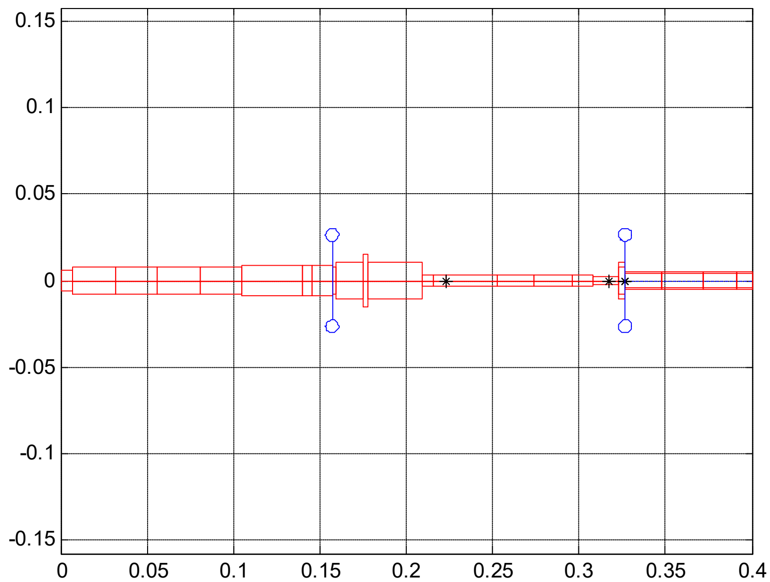

Fig. 2 Spindle of the textile machine.

4. Conclusions

Numerous numerical experiments were conducted, involving various values of random parameters governed by different probability density functions. Selected combinations will be demonstrated during the presentation. In line with the Central Limit Theorem, the results tend to converge towards the normal probability function as the number of RPs increases. The presented approach could be enhanced by incorporating more terms from Taylor's series into the eigenvalue expression. Based on the results of Monte Carlo simulations, the accuracy of the presented approach seems to be satisfactory.

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