30th International Conference
ENGINEERING MECHANICS 2024
Milovy, Czech Republic, May 14 – 16, 2024

SOLUTION METHODS FOR AN AEROELASTIC PROBLEM WITH COMBINED HARMONIC AND STOCHASTIC EXCITATION

Fischer C.*, **Náprstek J.****

Abstract: *Assessing responses in slender engineering structures facing both deterministic harmonic and stochastic excitation is often based on an approximation by the single-degree-of-freedom van der Pol-type nonlinear model. Determining the response probability density function involves solving the Fokker-Planck equation, which is generally a challenging task. Hence, semi-analytical and numerical methods come into play. This contribution reviews several possible techniques and spotlights the exponential-polynomial-closure method. The shown results are limited, as the paper reflects an early stage of the relevant research direction.*

Keywords: Fokker-Planck equation, stochastic averaging, numerical solution, Galerkin approximation, van der Pol-type oscillator, partial amplitudes, exponential-polynomial-closure method.

1. Introduction

Slender engineering structures, such as footbridges, masts, and power lines, are prone to excessive vibration. This tendency is particularly notable in the interplay between natural and excitation frequencies. In the lock-in region, where the structure's response stabilizes within a specific frequency range, it exhibits stationarity characterized by a dominant frequency and several superharmonic frequencies. Beyond this boundary, the response transforms into a non-stationary quasi-periodic state, with new frequencies emerging in a fan-shaped plot. Introducing additive random noise further complicates the scenario, the response process is stochastic. Understanding these dynamics is crucial for the design of slender structures, providing insights into their complex vibrational patterns and aiding in the development of effective mitigation strategies.

Exploring nonlinear dynamic systems under random excitation has long been an important subject with applications in various scientific and engineering domains. Researchers have developed analytical, semi-analytical, and numerical methods to obtain stationary PDFs or statistical moments, particularly focusing on systems influenced by Gaussian white noise. In contrast to the stationary response case, where usable solution procedures are often available, the non-stationary response case remains the subject of intensive research, presenting significant challenges even in scenarios limited to additive excitation.

The physical model utilized in this paper is the SDOF oscillator of the van der Pol type. This model is commonly used to depict transverse wind-generated vibrations under additive excitation, combining deterministic and random components. The normal form of this model is given as follows:

$$\dot{u} = v\,, \qquad \dot{v} = (\eta - \nu u^2)v - \omega_0^2 u + P\omega^2 \cos\omega t + h\xi(t)\,, \tag{1}$$

where: u, v are the displacement $[\mathrm{m}]$ and velocity $[\mathrm{ms^{-1}}]$; η, ν are the parameters of the linear and quadratic damping, respectively $[\mathrm{s^{-1}}, \mathrm{s^{-1}m^{-2}}]$; ω_0, ω are the eigen-frequency of the linear SDOF system and frequency of the vortex shedding $[\mathrm{s^{-1}}]$; $f(t)$ represents external excitation: $f(t) = P\omega^2\cos\omega t + h\xi(t)$; $P\omega^2$ and $\xi(t)$ are the amplitude of the harmonic excitation force $[\mathrm{ms^{-2}}]$ and the broadband Gaussian random process [*1*]; and h is the multiplicative constant $[\mathrm{ms^{-2}}]$.

* RNDr. Cyril Fischer, PhD.: Institute of Theoretical and Applied Mechanics, Prosecká 76, 190 00 Prague 9, fischerc@itam.cas.cz

** Ing. Jiří Náprstek, DSc.: Institute of Theoretical and Applied Mechanics, Prosecká 76, 190 00 Prague 9, naprstek@itam.cas.cz

doi: 10.21495/em2024-098

Using the Itô stochastic calculus, the response PDF is governed by the FPE:

$$\frac{\partial p(\mathbf{x},t)}{\partial t} = -\frac{\partial}{\partial x_j}\left(\kappa_j(\mathbf{x},t)\cdot p(\mathbf{x},t)\right) + \frac{1}{2}\frac{\partial^2}{\partial x_j \partial x_k}\left(\kappa_{jk}(\mathbf{x},t)\cdot p(\mathbf{x},t)\right). \tag{2}$$

Parameters $\kappa_j(\mathbf{x},t)$ and $\kappa_{jk}(\mathbf{x},t)$ represent the first and second derivative moments, generally referred to as drift and diffusion coefficients, respectively. In the stationary case, the left-hand side of Eq. (2) vanishes, resulting in a reduced FPE that is solvable in many particular cases (Lin and Cai, 1988). However, in the general non-stationary case, finding a solution for the complete FPE remains a challenging problem.

2. State of the art

According to Er (1998a), the solution of non-linear second order systems dates back to Kramers (1940), who presented solution of an one-dimensional undamped system with a non-linear stifness and an additive white noise excitation. Actually, the motivation in that paper was the first excursion problem in the theory of the velocity of chemical reactions, when the particle is originally caught in a potential hole but may escape in the course of time by passing over a potential barrier. If the function representing the potential barrier is smooth a reliable solution for any value of the viscosity is obtained.

The method of equivalent linearisation (Caughey, 1959), is the simplest solution method for non-linear systems with random excitation. Its approach can be regarded as the first approximation. In this reference the author studies the response of a nonlinear string to random excitation. It is shown that if the loading force is represented by truncated Gaussian white noise with uncorrelated Fourier coefficients, the mean squared deflection at every point is smaller than that for the equivalent linear string. The paper illustrates modification of the linearisation method, which is used for deterministic differential equations, for the stochastic case. The method of equivalent linearisation has been popular for over 65 years. However, there have been some missteps in its history, including an early conjecture by one of the pioneers that proved to be false, and an alternative to the standard procedure that went unrecognised for 27 years. Numerous reviews on method variants and applications are available; e.g. refer to the text by Elishakoff and Crandall (2016) and the papers cited therein.

Wen (1975) presents an approximate method for nonstationary solution of systems under random excitation, where the studied systems are supposed to include polynomial restoring force and the (filtered) shot noise type excitation. Using the Galerkin approach based on a time-dependent Hermite-series expansion, the Fokker-Planck equation (FPE) is is reduced to a system of first-order ordinary differential equations. Alternatively, when the excitation is non-white or when the FPE is difficult to solve, the perturbation method (Crandall, 1963) or statistical linearisation techniques (Caughey, 1959, 1963) are recommended, as reviewed also by Iwan and Yang (1972). Analogous procedure was used by the authors, (Náprstek and Fischer, 2024), and is illustrated by the numerical example in Chapt. 3.

Iyengar and Dash (1978) propose a technique which is capable to adopt the non-Gaussian excitation. Their approach belongs to the class of closure techniques. The individual methods in this class differ by different assumptions made about the statistical structure of the response. Even in cases where the response may be non-Gaussian, it would be possible to find a function of the response which can be approximated in terms of the Gaussian distribution via error minimization. The method due to Iyengar and Dash (1978), the Gaussian closure technique, automatically leads to an associated linear system driven by a Gaussian input. This, in fact, implies similarity of the method with the statistical linearisation techniques.

When the system is highly non-linear, or when multiplicative random excitations are present, i.e., when the probability distribution of the system response is far from being Gaussian, more response moments than two has to be approximated. This generalization leads to non-Gaussian closure methods as that used by Assaf and Zirkle (1976). In this highly instructive paper authors approximate the PDF of the response via the Edgeworth-type expansion. It is a rearrangement of the Gram-Charlier expansion so that the accuracy increases with the natural order of the terms (Cramér, 1946). The PDF is assumed in the form

$$p(x) = p_0(x)\left[1 + \frac{1}{3!}\frac{\lambda_3}{\sigma_x^3} H_3\left(\frac{x-m_x}{\sigma_x}\right) + \frac{1}{4!}\frac{\lambda_4}{\sigma_x^4} H_4\left(\frac{x-m_x}{\sigma_x}\right) + \frac{10}{6!}\frac{\lambda_3^2}{\sigma_x^6} H_6\left(\frac{x-m_x}{\sigma_x}\right) + \ldots\right] \tag{3}$$

where $p_0(x)$ is the Gaussian PDF and m_x, σ_x are the mean value and the standard deviation of x, $H(\cdot)$ are the Chebychev-Hermite polynomials, and λ_i the ith-order semi-invariants. The authors claim that four terms in the expansion are usually sufficient. On the other hand, the obtained series can lead to negative probabilities as pointed out by Grigoriu (1991).

Cai and Lin (1988) and Cai et al. (1992) have developed a new approximation procedure in which a given non-linear system is replaced by another non-linear system belonging to the class of generalized stationary potential (Lin and Cai, 1988) for which the exact stationary solutions are obtainable. The replacement is based on the premise of preservation of the average energy dissipation in the replacing and the original systems.

For weakly non-linear systems with weak excitations, the stochastic average method, (Roberts and Spanos, 1986), represents a powerful alternative. The perturbation method, due to Crandall (1963), belongs to this category as well. For systems with random additive excitation, moment equations derived from the Itô's derivative rule can be used to compute the statistical moments instead of the FPE. Introduction of the central-moment closure or cumulant-neglect closure to the moment equations creates the hierarchy of equations limited to a desired level, (Wu and Lin, 1984).

The exponential-polynomial-closure (EPC) method was initially published by Er (1998b). In the original stationary setting, it assumes the sought PDF of an approximate solution in the form of an exponential polynomial:

$$p(\mathbf{x};\mathbf{a}) = C\exp\left(Q_n(\mathbf{x};\mathbf{a})\right) \tag{4}$$

Here, $\mathbf{x}$ is the state vector, $\mathbf{a}$ is the unknown parameter vector, and $Q_n(\mathbf{x};\mathbf{a})$ is a polynomial function. The algebraic system for the unknown parameters $\mathbf{a}$ results from the Galerkin approximation with respect to basis functions $h_k(\mathbf{x}) = x_1^{k_1}\dots x_{n_x}^{k_{n_x}} f_N(\mathbf{x})$, where $k = k_1 + \ldots + x_{n_x}$ and f_N is the PDF solution using Gaussian closure.

Since then, variants of the EPC method have been proposed for different settings of the stationary PDF solutions of nonlinear stochastic oscillators. Modifications for the non-linear, non-stationary case have only recently emerged, implicitly allowing for non-Gaussian excitation (Guo et al., 2020). A further modification by Wang et al. (2023) claims superior performance with respect to smaller errors at the PDF tails compared to the results of Monte-Carlo simulations.

3. Numerical example

The response of the van der Pol oscillator in Eq. (1) is stationary in the lock-in region, which corresponds to interval $\omega \in (0.85, 1.35)$ with respect to natural frequency $\omega_0 = 1$, parameters $\eta = 1/2$, $\nu = 1/4$, $P = 1$, and the stochastic parameters $h = 1$, $S = 1$. In order to use the stochastic averaging method, (Náprstek and Fischer, 2024), the response variables are expressed in the trigonometric form as follows

$$u(t) = a_c\cos\omega t + a_s\sin\omega t\,,\quad v(t) = -a_c\omega\sin\omega t + a_s\omega\cos\omega t\,,\quad \dot{a}_c\cos\omega t + \dot{a}_s\sin\omega t = 0\,.$$

New parameters a_c, a_s are random variables representing the *partial amplitudes* of the response. In the stationary case, they are described by the reduced FPE. It is solvable analytically for zero detuning ($\Delta = 0$) between excitation and natural frequency. For $0 < \Delta$, the solution was sought using the Galerkin approach (Náprstek and Fischer, 2024). The result is illustrated in Fig. 1. PDFs with respect to partial amplitudes a_c, a_s is shown for $M = 2$ and a non-negligible value of detuning $\delta = 0.10$. The contour plot of the estimated cross-PDF $p(a_c, a_s)$ is shown on the left. The middle plot depicts the sections of the PDF for fixed values $a_c = \{-3/2.0, 3/2\}$ and the right-hand plot illustrates the sections for the selected values $a_s = \{2, 3, 4\}$. The sections and the corresponding colors are indicated as horizontal/vertical lines in the left-hand plots. The dashed curves show the basic analytical solution which is valid for the case zero no

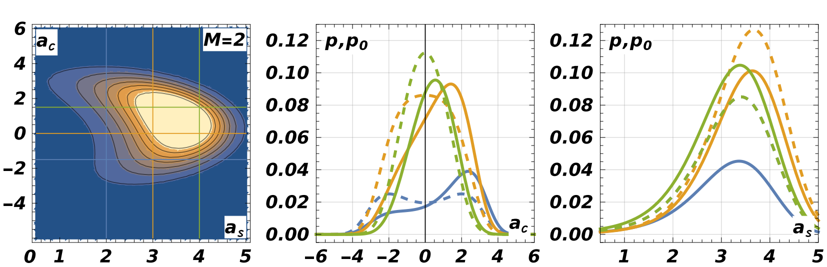

Fig. 1: The Galerkin approximation of the stationary cross-PDF for $M = 2$ (number of stochastic moments) and detuning value $\Delta = 0.10$.

detuning is assumed. The estimates including the $M = 2$ Galerkin approximations are shown in solid. It can be confirmed than corrections for $M > 2$ do not bring visible improvement.

4. Concluding remarks

The study of nonlinear dynamic systems under random excitations is an attractive and widely applicable topic in various scientific and engineering domains. Nonlinearity in the mathematical model can result in a non-zero mean of the response, even when the excitation mean is zero. As the PDFs obtained by different methods are only approximative, their behaviours at the tail positions poses a challenging problem. This contribution, based on a just started research, presented a historical and state-of-the-art review of available methods and some preliminary results regarding the stationary nonlinear state based on the stochastic averaging and subsequent Galerkin approach. However, the final target is the fully non-linear non-stationary case, which is still subject of an ongoing area of research.

Acknowledgment

The kind support of Czech Science Foundation project No. 24-13061S and of the RVO 68378297 institutional support are gratefully acknowledged.

References

Assaf, S. H. A. and Zirkle, L. D. (1976) Approximate analysis of non-linear stochastic systems. *Int. J. Control*, 23, 4, pp. 477–492.

Cai, G. Q. and Lin, Y. K. (1988) A new approximate solution technique for randomly excited non-linear oscillators. *Int. J. Non Linear Mech.*, 23, 5–6, pp. 409–420.

Cai, G. Q., Lin, Y. K., and Elishakoff, I. (1992) A new approximate solution technique for randomly excited non-linear oscillators–II. *Int. J. Non Linear Mech.*, 27, 6, pp. 969–979.

Caughey, T. K. (1959) Response of a nonlinear string to random loading. *J. Appl. Mech.*, 26, 3, pp. 341–344.

Caughey, T. K. (1963) Equivalent linearization techniques. *J. Acoust. Soc. Am.*, 35, 11, pp. 1706–1711.

Cramér, H. (1946) *Mathematical Methods of Statistics*. Princeton University Press, Princeton.

Crandall, S. H. (1963) Perturbation techniques for random vibration of nonlinear systems. *J. Acoust. Soc. Am.*, 35, pp. 1700–1705.

Elishakoff, I. and Crandall, S. H. (2016) Sixty years of stochastic linearization technique. *Meccanica*, 52, 1–2, pp. 299–305.

Er, G. K. (1998a) Multi-Gaussian closure method for randomly excited non-linear systems. *Int. J. Non Linear Mech.*, 33, 2, pp. 201–214.

Er, G. K. (1998b) A new non-Gaussian cloasure method for the PDF solution of non-linear random vibrations. In: *Proceedings of The 12th ASCE Engineering Mechanics Conference*, La Jolla, California. ASCE, Reston, VA, pp. 1403–1406.

Grigoriu, M. (1991) A consistent closure method for non-linear random vibration. *Int. J. Non Linear Mech.*, 26, 6, pp. 857–866.

Guo, S. S., Shi, Q., and Xu, Z. D. (2020) Stochastic responses of nonlinear systems to nonstationary non-Gaussian excitations. *Mech. Syst. Sig. Process.*, 144, pp. 106898.

Iwan, W. D. and Yang, I. M. (1972) Application of statistical linearization techniques to nonlinear multidegree-of-freedom systems. *J. Appl. Mech.*, 39, 2, pp. 545–550.

Iyengar, R. N. and Dash, P. K. (1978) Study of the random vibration of nonlinear systems by the Gaussian closure technique. *J. Appl. Mech.*, 45, 2, pp. 393–399.

Kramers, H. A. (1940) Brownian motion in a field of force and the diffusion model of chemical reactions. *Physica*, VII, 4, pp. 284–304.

Lin, Y. K. and Cai, G. Q. (1988) Equivalent stochastic systems. *J. Appl. Mech.*, 55, 4, pp. 918–922.

Náprstek, J. and Fischer, C. (2024) Averaging-based characteristics of the response induced by combined random and harmonic excitation. In: *Proceedings of the 15th International Conference on Vibration Problems. ICoVP 2023*, Singapore. Springer, Lecture Notes in Mechanical Engineering. in print.

Roberts, J. B. and Spanos, P. D. (1986) Stochastic averaging: An approximate method of solving random vibration problems. *Int. J. Non Linear Mech.*, 21, 2, pp. 111–134.

Wang, K., Wang, J., Jia, S., Zhu, Z., Yu, Z., and Xu, L. (2023) Non-stationary nonzero mean probabilistic solutions of nonlinear stochastic oscillators subjected to both additive and multiplicative excitations. *Chin. J. Phys.*, 81, pp. 64–77.

Wen, Y. K. (1975) Approximate method for nonlinear random vibration. *Journal of the Engineering Mechanics Division*, 101, 4, pp. 389–401.

Wu, W. F. and Lin, Y. K. (1984) Cumulant-neglect closure for non-linear oscillators under random parametric and external excitations. *Int. J. Non Linear Mech.*, 19, 4, pp. 349–362.