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NON-STATIONARY WAVES IN THICK ELASTIC AND VISCOELASTIC PLATES

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Abstract: *This study deals with the dynamics of non-stationary wave propagation in thick, homogeneous, isotropic, elastic and viscoelastic plates, employing a combination of analytical and numerical approaches. The first aim of this work is to efficiently enumerate the previously derived solution for transient wave propagation in a thick elastic plate. For this purpose, a numerical inverse Laplace transform (NILT) algorithm is used, which significantly improves computational efficiency. The results from this semi-analytical approach were compared to those results obtained using a Finite Element (FE) model to ensure the accuracy and validity of both methods. Additionally, the effect of viscoelasticity on wave propagation characteristics is discussed in this work.*

Keywords: Wave propagation, thick elastic plate, thick viscoelastic plate, analytical solution, numerical solution.

1. Introduction

In this work, non-stationary waves in a thick elastic and viscoelastic plate are investigated using a numerical and analytical approach. In contrast to the numerical solution, the knowledge of the analytical one can be advantageously used in solving many inverse problems, especially due to its computational speed. This is used, for example, to identify material properties, the source of excitation and to detect defects and factors that could lead to a limited life and reliability of machines. One could find the application of this approach in civil, aeronautical, mechanical and many other engineering fields.

The problem of wave propagation in an elastic thick plate was solved in the work of Valeš (1983), where the solution for basic mechanical quantities in the Laplace domain was derived. The evaluation of the derived formulas was then carried out in Pátek (1996) using the exact inverse Laplace transform based on the dispersion curves calculation and on the use of the residue theorem. This approach is exact, but with respect to high demands on CPU time, it is unsuitable for effective solving of inverse problems.

In view of this fact, the exact analytical inversion procedure will be replaced by the numerical inverse Laplace transform (NILT), specifically using the algorithm presented in Brančík (1999), which is based on the combination of FFT and Wynn's algorithm (Cohen , 2007). The same algorithm proved to be suitable for solving similar problems of transient waves as shown, e.g. in Šulda (2024). As known, every real material shows some degree of damping, i.e. energy dissipation, and therefore this work will also focus on the attenuation effects of the viscoelastic material on waves propagated in the plate. The discussion of this effect will be based on a comparison with the elastic case.

2. Analytical solution for a thick elastic plate

An infinite thick plate of thickness $2d$ is considered. The plate is assumed to be in a cylindrical coordinate system with $r \in \langle 0; \infty \rangle$ as the radial coordinate, $\vartheta \in \langle 0; 2\pi \rangle$ as the angular coordinate, and $z \in \langle -d; d \rangle$

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as the coordinate in the plate thickness direction. On the upper surface, i.e. $z = -d$, the plate is excited by a constant transverse pressure of magnitude σ_0 on a circular region of radius R. The remaining parts of both plate surfaces are considered free of load. In such a case, the boundary conditions can be formulated as follows

$$\begin{aligned} \sigma_z(r,-d,t) &= \begin{cases} -\sigma_0 & \text{for } r < R \\ 0 & \text{otherwise} \end{cases}, \qquad & \tau_{rz}(r,-d,t) = 0, \\ \sigma_z(r,d,t) &= 0, & \tau_{rz}(r,d,t) = 0, \end{aligned} \tag{1}$$

where the functions σ_z and τ_{rz} represent the normal stress in the z direction and the shear stress in the rz plane, respectively. The equations of motion for such a plate can be derived using the Cauchy equations formulated in the cylindrical coordinate system (Graff , 1991). Due to the rotational symmetry of the problem, these equations will be independent of the coordinate ϑ. Substituting the kinematic equations for small strains into the constitutive relations and then into the Cauchy equations, the following equations for dilatation Δ and rotation ω_ϑ can be obtained (Valeš, 1983)

$$\begin{aligned} \frac{\partial^2 \Delta}{\partial t^2} &= c_1^2 \left(\frac{\partial^2 \Delta}{\partial r^2} + \frac{\partial^2 \Delta}{\partial z^2} + \frac{1}{r}\frac{\partial \Delta}{\partial r} \right), \\ \frac{\partial^2 \omega_\vartheta}{\partial t^2} &= c_2^2 \left(\frac{\partial^2 \omega_\vartheta}{\partial r^2} + \frac{\partial^2 \omega_\vartheta}{\partial z^2} + \frac{1}{r}\frac{\partial \omega_\vartheta}{\partial r} - \frac{\omega_\vartheta}{r^2} \right), \end{aligned} \tag{2}$$

where

$$\Delta = \frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}, \;\; \omega_\vartheta = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right). \tag{3}$$

As clear, the system of partial differential equations (2) for Δ and ω_ϑ is uncoupled in this case, which significantly simplifies the solving procedure. The constants $c_1 = \sqrt{(\lambda + 2G)/\rho}$ and $c_2 = \sqrt{G/\rho}$ denote the phase velocity of dilatation and shear waves in an elastic continuum of the density ρ, the shear modulus G and the Lame's constant λ. The solution of the system (2) can be advantageously found using the Laplace transform in time t and the Hankel transform in the spacial coordinate r. The resulting relations for the Laplace transforms of displacements $\overline{u}_r$ and $\overline{u}_z$ can be written as (Valeš, 1983)

$$\begin{aligned} \overline{u}_r &= \frac{\sigma_0 R}{2G} \int_0^\infty \left(\frac{F_3}{pL} - \frac{G_3}{pT} \right) \frac{1}{\gamma} J_1(\gamma R) J_1(\gamma r)\, d\gamma, \\ \overline{u}_z &= \frac{\sigma_0 R}{2G} \int_0^\infty \left(\frac{G_4}{pT} - \frac{F_4}{pL} \right) \frac{K_1}{\gamma} J_1(\gamma R) J_0(\gamma r)\, d\gamma, \end{aligned} \tag{4}$$

where the real parameter γ is the variable of the Hankel transform, and p represents the complex variable of the Laplace transform. The functions J_0 and J_1 are the Bessel functions of the first kind and of the zero and first order. The remaining complex functions F_3, F_4, G_3, G_4, L, T, K_1 introduced in (4) are mostly defined as a combination of hyperbolic sines and cosines and can be found in Valeš (1983).

Considering the relations for the Laplace transforms of time derivatives and with respect to the zero initial condition of the solved problem, one can write $\overline{v}_r = p\overline{u}_r$, $\overline{v}_z = p\overline{u}_z$ for the transforms of velocities. The formulas for displacement and velocity components were evaluated using a Matlab code. A plate with the thickness of $2d = 40$ mm was considered and excited with the constant pressure $\sigma_0 = 1$ MPa at a circular area with the radius $R = 2$ mm. The material parameters of the plate corresponded to steel, i.e. $\rho = 7\,800$ kg/m^3, $E = 2.11 \cdot 10^{11}$ Pa and $\nu = 0.3$. The resulting visualisation of wavefronts in one half of the plate cross-section at times of 5, 10 and 20 μs is depicted using the velocity component v_z in Fig. 1. The vertical axis corresponds to the radial direction r and the horizontal one to the vertical coordinate z. Different types of waves propagated in the plate by their characteristic velocities can be simply identified from Fig. 1. The fastest is the dilatational wave (P-wave), which propagates at speed $c_1 \doteq 6\,020$ m/s and which is followed by a shear wave (S-wave) propagated with $c_2 \doteq 3\,218$ m/s. The largest amplitude gains the Rayleigh wave (R-wave), which travels on the free surface of the plate, and its speed can be estimated as $c_R \approx 0.92 c_2 \doteq 2\,960$ m/s.

3. Comparison of semi-analytical results with results obtained by FEM

Regarding the fact that the analytical formulas are complex, it is advisable to use another method to verify the evaluation procedure. For this purpose, an axisymmetric problem of a thick elastic plate of thickness

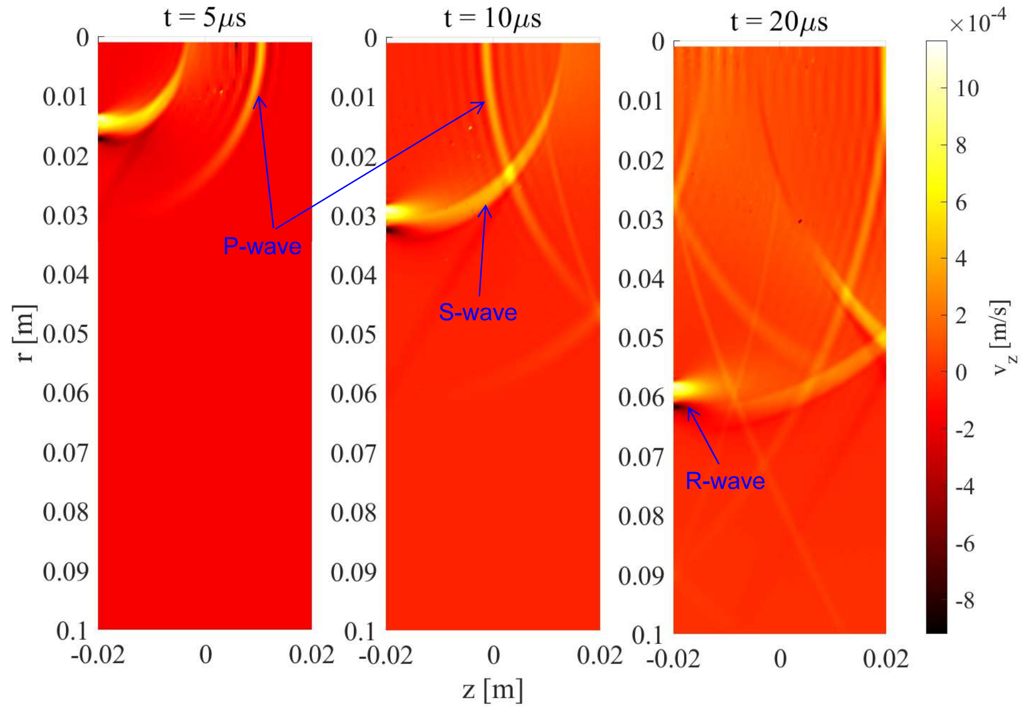


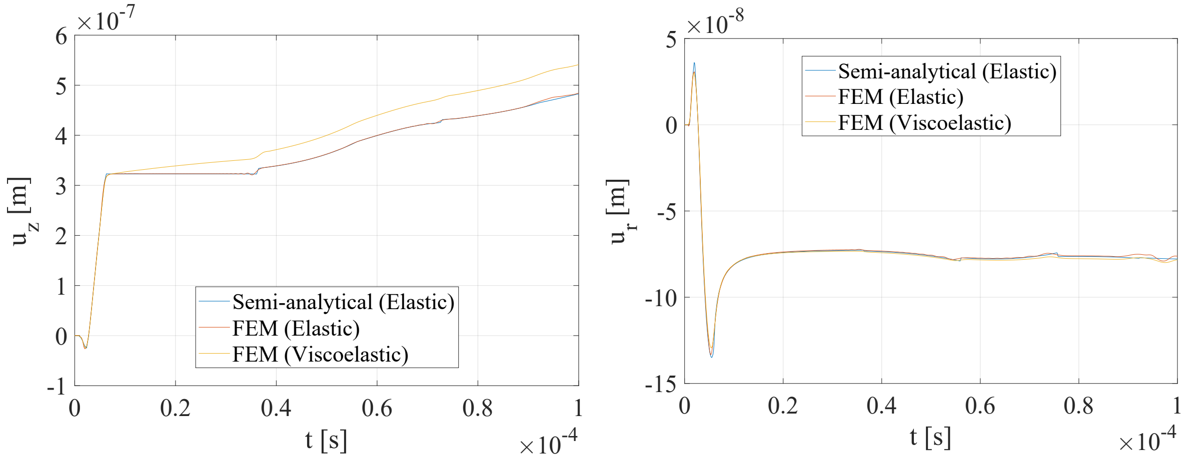
Fig. 1: Velocity v_z distribution in a thick steel plate.

40 mm and radius 100 mm was solved in the software MSC.Marc. The linear axisymmetric elements (type 10) with the size 0.4×0.4 mm were used. To study the effect of energy dissipation, the viscoelastic plate was also considered. The Young moduli of both plates correspond to polypropylene (PP) and were chosen as $E = 2.8090 \cdot 10^9$ Pa (see Šulda (2024)). The viscoelastic properties of the second plate was modelled using the standard viscoelastic solid in Zener configuration, the parameters of which were identified for a thin PP rod in Šulda (2024). These parameters are summarised in Tab. 1. With respect to the maximal wave speed in these materials ($c_1 = 2\,203$ m/s) and to the mentioned size of elements, the integration time step of the Newmark integration method was chosen as $1.82 \cdot 10^{-7}$ s.

ρ [kg/m^3]	ν [-]	E_E [Pa]	E_1 [Pa]	λ_1 [Pa·s]
928.6	0.35	$2.2608 \cdot 10^9$	$5.4821 \cdot 10^8$	$1.9099 \cdot 10^4$

Tab. 1: Material parameters of PP modelled by the standard viscoelastic solid (Zener model).

Fig. 2a) shows a comparison of the displacement u_z for the $z = -20$ mm and $r = 4$ mm from both approaches and for elastic and viscoelastic material up to time $t_{max} = 100$ μs. The compared results for the elastic case have a maximum relative error of 1.1 %. Approximately from $t = 88$ μs, the numerical results are influenced by the P-waves reflected from the plate boundary at $r = 100$ mm. On the other hand, the effect of the finite dimension of the plate in the transverse direction z has a more significant impact in both cases after the time of 35 μs (approx. $4d/c_1$) when the P-wave reflected from the bottom surface of the plate arrived at the monitored point. After the R-wave passes this point (approx. 6.5 μs) and before the arrival of the mentioned reflected P-wave, it can be seen that the excitation does not cause any change in u_z in the elastic case. In the viscoelastic case, a rising trend of u_z occurs when constant stress is applied, and thus, there is a remarkable deviation from the results for the elastic plate. Another change-point in the results is at 75 μs. This time corresponds to the arrival of the reflected S-wave. These same changes in solutions can be observed in Fig. 2b), where the displacement u_r is shown at the same monitored location. The results of the numerical and the semi-analytical elastic model deviate by the maximal relative error 3 % up to t_{max}, except the initial arrival of the wave. In this region, the maximal relative error is 14 % in peaks. Within this time interval, we can also observe the effect of energy dissipation resulting in a small decrease in amplitudes. The viscoelastic case differs relatively from the elastic one in the mentioned peaks by 3 %.



a) displacement u_z *b) displacement u_r*

Fig. 2: Comparison of semi-analytical and numerical results obtained for elastic and viscoelastic plate.

4. Conclusions

The analytical solution for the non-stationary state of stress in a thick elastic plate under transverse pressure loading previously derived in Valeš (1983) was evaluated using a numerical inverse Laplace transform algorithm. Contrary to the traditional and exact evaluation procedure based on dispersion curves and residual theorem, this approach enables the effective calculation of plate response in the 2D domain. These semi-analytical results agreed well with the results obtained by FE simulation performed in the software MSC.Marc. The FE model was also used to study the effect of energy dissipation in short times.

This work is a starting point for the derivation of the analytical solution for (i) a similar problem of a thick viscoelastic plate and (ii) a problem of a thick layered viscoelastic plate. The latter one could then be used as an approximation of the response of a viscoelastic plate made of functionally graded material.

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